# 第7章 第一次习题课

## 一、多元函数的极限

### 1、求极限

方法：利用多元初等函数的连续性、极限的运算性质、极限存在准则（夹逼准则）、重要极限、等价无穷小替换、化为一元函数的极限等。注意各种方法的综合运用。

### 2、判别极限不存在.

方法：取两种不同的方式极限不相等则极限不存在

例1 $\lim\limits_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)^{xy}$.  （$0^0$ 型）

注：$u(x)^{v(x)}=e^{v(x)\ln u(x)}$

解法一 先求 $\lim\limits_{\substack{x\to 0\\ y\to 0}} xy\ln(x^2+y^2)$

$$0\le |xy\ln(x^2+y^2)|\le \frac{1}{2}|(x^2+y^2)\ln(x^2+y^2)|$$

$$\lim_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)\ln(x^2+y^2) \xlongequal{x^2+y^2=t} \lim_{t\to 0^+} t\ln t=0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} xy\ln(x^2+y^2)=0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)^{xy}=e^{\lim\limits_{\substack{x\to 0\\ y\to 0}} xy\ln(x^2+y^2)}=e^0=1$$

$$\lim_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)^{xy}.$$

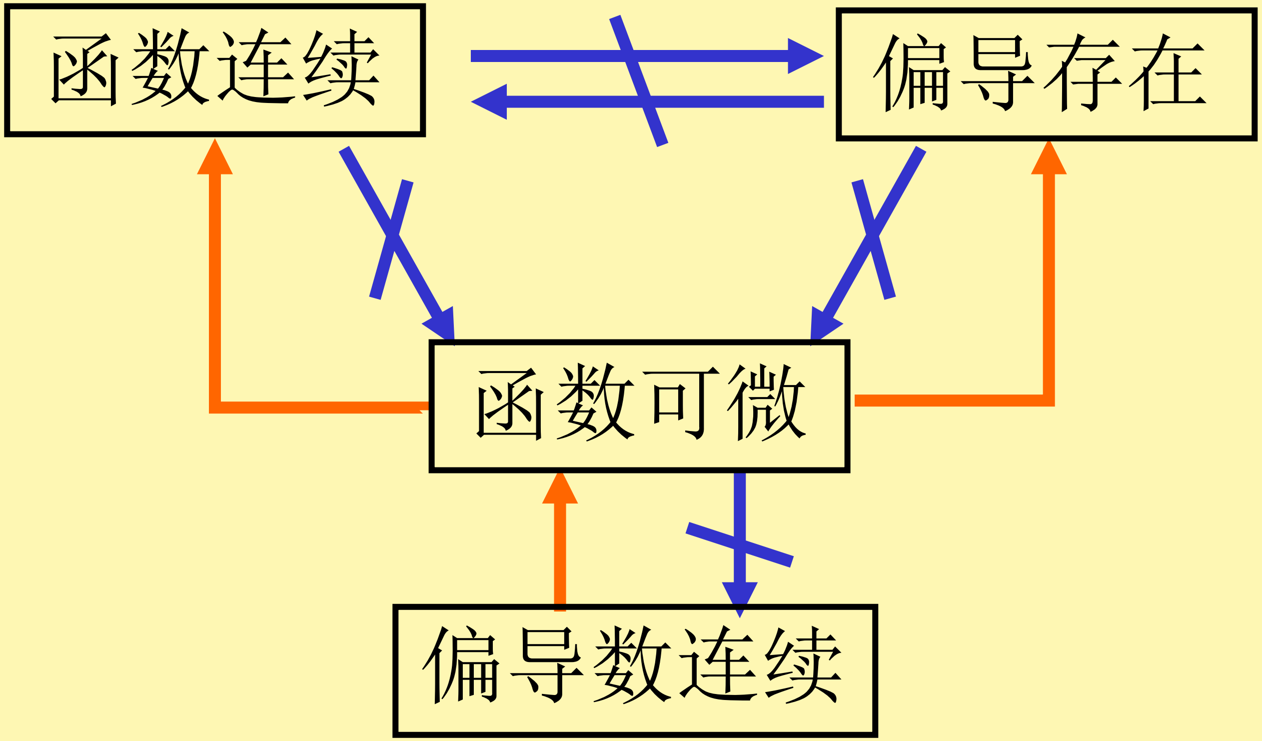
解法二

$$\lim_{\substack{x\to 0\\ y\to 0}} xy\ln(x^2+y^2)=\lim_{\rho\to 0}2\rho^2\cos\theta\sin\theta\ln\rho$$

$$\lim_{\rho\to 0}2\rho^2\ln\rho=0, |\cos\theta\sin\theta|\le 1$$

$$\Rightarrow 原极限 = e^{\lim\limits_{\substack{x\to 0\\ y\to 0}} xy\ln(x^2+y^2)}=1$$

## 二、多元函数的连续、偏导存在、可微性的讨论




例、$f(x,y)$在$(x_0,y_0)$处连续是函数$f(x,y)$在$(x_0,y_0)$处可微的__必要__条件。

例2

(1) 设$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$,

则在(0,0) ___**B**___.

（A）不连续　（B）偏导存在　(C）可微

解:
$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{\sqrt{x^2+y^2}}=\lim_{\rho\to 0}\rho\cos\theta\sin\theta=0=f(0,0)$$

所以，$f(x,y)$在(0,0)处连续

$$f_x'(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0=f_y'(0,0)$$

所以，$f(x,y)$在(0,0)处偏导存在

例2

(1) 设$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$,

则在(0,0) ___ *B* ___.

（A）不连续　　（B）偏导存在　　(C) 可微

$$\text{可微}\Leftrightarrow \Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$$
$$=A\Delta x+B\Delta y+o(\rho)=f_x\Delta x+f_y\Delta y+o(\rho)$$

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\because \quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta z-f_x'(0,0)\Delta x-f_y'(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\dfrac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2} \quad \because \quad \lim_{\substack{k\Delta x=\Delta y\\ \Delta x\to 0}}\frac{k\Delta x^2}{(1+k^2)\Delta x^2}=\frac{k}{1+k^2}$$

$$\therefore \quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta z-f_x'(0,0)\Delta x-f_y'(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}\text{不存在}$$

所以，$f(x,y)$在$(0,0)$处不可微

例2 $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

在(0,0)处（1）偏导是否存在？（2）可微？

（3）偏导连续？

**解**

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x^2\sin\dfrac{1}{\Delta x^2}}{\Delta x}=0$$

同理$f_y(0,0)=0$

例2 $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

在(0,0)处（2）可微?

$$f_x(0,0)=f_y(0,0)=0$$

$$\because \quad \lim_{\rho\to 0}\frac{\Delta z-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\rho}$$

$$=\lim_{\rho\to 0}\frac{(\Delta x^2+\Delta y^2)\sin\dfrac{1}{\Delta x^2+\Delta y^2}}{\rho}=\lim_{\rho\to 0}\frac{\rho^2\sin\dfrac{1}{\rho^2}}{\rho}=0$$

$\therefore f(x,y)$在(0,0)处可微

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$ 偏导连续？

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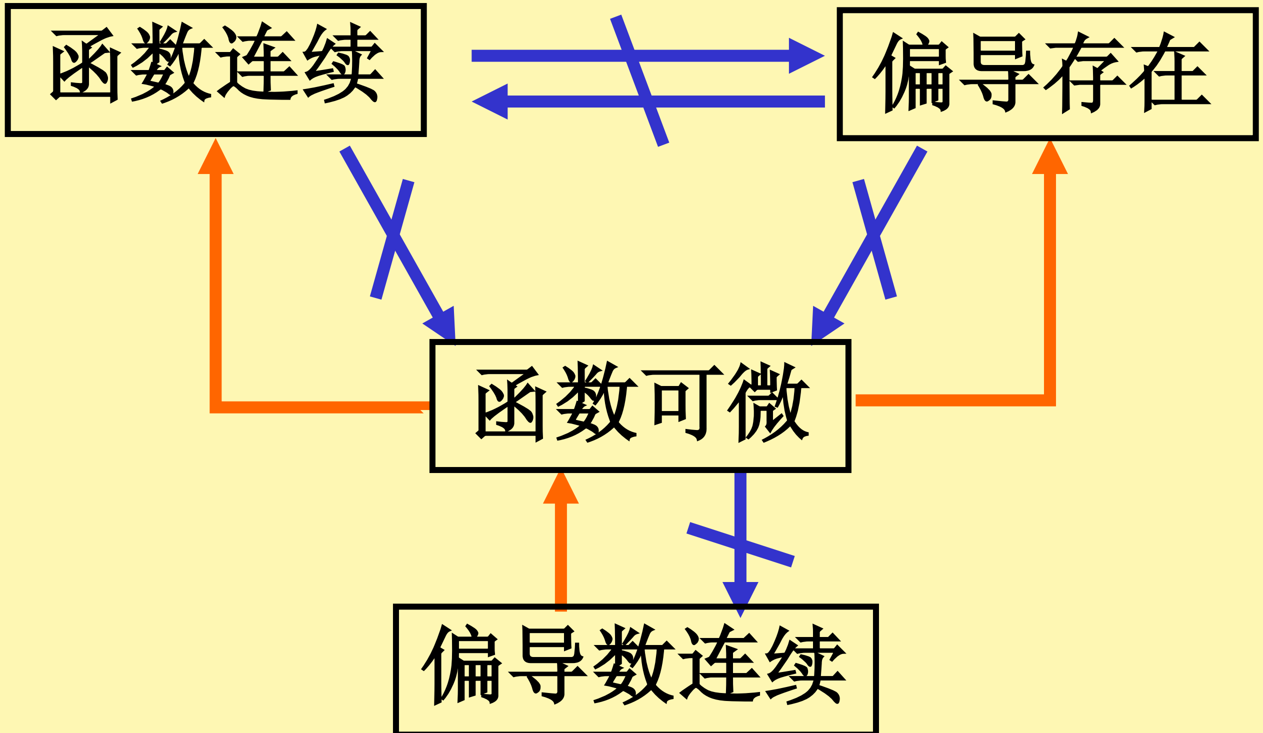
$$f_x(x,y)=\begin{cases}2x\sin\dfrac{1}{x^2+y^2}-\dfrac{2x}{x^2+y^2}\cos\dfrac{1}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\lim_{\substack{y\to 0\\ x\to 0}} f_x(x,y)=f_x(0,0)\,?$$

$$\lim_{\substack{y=x\\ x\to 0}} f_x(x,y)=\lim_{\substack{y=x\\ x\to 0}}\left(2x\sin\frac{1}{2x^2}-\frac{1}{x}\cos\frac{1}{2x^2}\right)$$ 不存在

$\therefore f_x(x,y)$在$(0,0)$处不连续 同理$f_y(x,y)$不连续

# 多元函数连续、可导、可微的关系




$$f(x,y)=\begin{cases}\left(x^2+y^2\right)\sin\left(\dfrac{1}{x^2+y^2}\right), & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

在原点可微，但偏导不连续

## 三、求多元具体函数的偏导数、高阶偏导数、全微分

例3 (1) $z=\arctan\frac{y}{x}$, $z_{xy}''=$ $\underline{\dfrac{y^2-x^2}{(x^2+y^2)^2}}$

解

$$\frac{\partial z}{\partial x}=\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}=\frac{-y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}=\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

(2) $z=\arctan\frac{y}{x}$, $dz\big|_{(1,1)}=$ $\underline{-\frac{1}{2}dx+\frac{1}{2}dy}$

## 四、求多元抽象函数的偏导数、高阶偏导数、全微分

例4 (1)设$z=f(xy,\frac{x}{y})+g(x^2+y^2)$，$g$二阶可导，$f$具有二阶连续偏导，求$\frac{\partial^2 z}{\partial x\partial y}$.

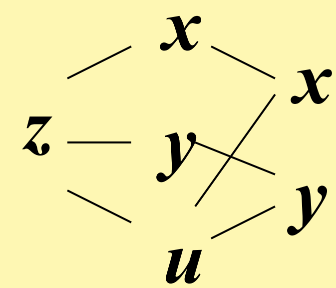
解 $$\frac{\partial z}{\partial x}=yf_1+\frac{1}{y}f_2+2xg'$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1+y(xf_{11}-\frac{x}{y^2}f_{12})-\frac{1}{y^2}f_2+\frac{1}{y}(xf_{21}-\frac{x}{y^2}f_{22})$$

$$+4xyg''$$

$$=f_1'+xyf_{11}''-\frac{1}{y^2}f_2'-\frac{x}{y^3}f_{22}''+4xyg''$$

(2) 设$z=f(x, y, u)$，$u=xe^y$，$f$ 具有二阶连续偏导数，求 $\dfrac{\partial^2 z}{\partial x\partial y}$




解 $\dfrac{\partial z}{\partial x} = f_1' + f_3' \cdot \dfrac{\partial u}{\partial x} = f_1' + e^y f_3',$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial y}(f_1' + e^y f_3') = \frac{\partial f_1'}{\partial y} + e^y f_3' + e^y \frac{\partial f_3'}{\partial y}$$

$$= f_{12}'' + f_{13}'' \cdot \frac{\partial u}{\partial y} + e^y f_3' + e^y (f_{32}'' + f_{33}'' \cdot \frac{\partial u}{\partial y})$$

$$= f_{12}'' + f_{13}'' \cdot xe^y + e^y f_3' + e^y (f_{32}'' + f_{33}'' \cdot xe^y)$$

$$= f_{12}'' + xe^y f_{13}'' + e^y f_3' + e^y f_{32}'' + xe^{2y} f_{33}''$$

(3) 设函数$f(x,y)$可微，$f(1,1)=1, f_x'(1,1)=a,$
$f_y'(1,1)=b,$又记$\varphi(x)=f(x,f(x,x)),$求$\varphi(1),\varphi'(1).$

解 $\varphi(1)=f(1,f(1,1)) \quad =f(1,1)=1$

$$\varphi'(x)=f_x'(x,f(x,x))+f_y'(x,f(x,x))\frac{df(x,x)}{dx}$$

$$=f_x'(x,f(x,x))+f_y'(x,f(x,x))[f_x'(x,x)+f_y'(x,x)]$$

$$\varphi'(1)=f_x'(1,f(1,1))+f_y'(1,f(1,1))[f_x'(1,1)+f_y'(1,1)]$$

$$=f_x'(1,1)+f_y'(1,1)[f_x'(1,1)+f_y'(1,1)]$$

$$=a+b[a+b]=a+ab+b^2$$

## 五、隐函数的偏导数、全微分的计算

例5 (1).由方程$xyz+\sqrt{x^2+y^2+z^2}=0$确定隐含数$z=z(x,y)$，求$dz\big|_{(1,0,-1)}$

解 方程两边对$x,y$求偏导

$$yz+xy\frac{\partial z}{\partial x}+\frac{1}{\sqrt{x^2+y^2+z^2}}(x+z\frac{\partial z}{\partial x})=0$$

$$xz+xy\frac{\partial z}{\partial y}+\frac{1}{\sqrt{x^2+y^2+z^2}}(y+z\frac{\partial z}{\partial y})=0$$

将$x=1,y=0,z=-1$代入得

$$\frac{\partial z}{\partial x}\Big|_{(1,0,-1)}=1 \quad \frac{\partial z}{\partial y}\Big|_{(1,0,-1)}=-\sqrt{2}$$

$$dz\big|_{(1,0,-1)}=dx-\sqrt{2}dy$$

(2) $u=\sin(y+3z)$，$z$由$z^2y-xz^3-1=0$确定，

求$\left.\dfrac{\partial u}{\partial x}\right|_{\substack{x=1\\y=0}}=\cos 3$

解：$$\frac{\partial u}{\partial x}=3\cos(y+3z)\cdot\frac{\partial z}{\partial x}=3\cos(y+3z)\frac{z^3}{2yz-3xz^2}$$

$z^2y-xz^3-1=0$确定$z=z(x,y)$

方程两边对$x$求偏导：$y\cdot 2z\cdot\dfrac{\partial z}{\partial x}-z^3-3xz^2\cdot\dfrac{\partial z}{\partial x}=0$

$$\frac{\partial z}{\partial x}=\frac{z^3}{2yz-3xz^2}$$

(3)设$\frac{1}{z}-\frac{1}{x}=f(\frac{1}{y}-\frac{1}{x})$, $f(u)$可微, 求 $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}$

解： 两端求对$x$的偏导数，得

$$-\frac{1}{z^2}\cdot\frac{\partial z}{\partial x}+\frac{1}{x^2}=f'(u)\cdot\frac{1}{x^2}$$

两端同乘以$x^2z^2$： $z^2-x^2\frac{\partial z}{\partial x}=z^2f'(u)$ (1)

两端求对$y$的偏导数：$-\frac{1}{z^2}\cdot\frac{\partial z}{\partial y}=f'(u)\cdot(-\frac{1}{y^2})$

两端同乘以$y^2z^2$： $-y^2\frac{\partial z}{\partial y}=-z^2f'(u)$ (2)

(1)式+(2)式 即得$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2$

**(4)** 设$F(x+\frac{z}{y}, y+\frac{z}{x})=0$, $F$可微，求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, dz$.

解法一：利用公式，令：

$$\varphi(x, y, z)=F(x+\tfrac{z}{y}, y+\tfrac{z}{x})$$

$$\varphi_x = F_1 + F_2\cdot(-\frac{z}{x^2}) \qquad \varphi_y = F_1\cdot(-\frac{z}{y^2}) + F_2$$

$$\varphi_z = F_1\cdot\frac{1}{y} + F_2\cdot\frac{1}{x}$$

$$\frac{\partial z}{\partial x} = -\frac{\varphi_x}{\varphi_z} = \frac{-F_1+\frac{z}{x^2}F_2}{\frac{1}{y}F_1+\frac{1}{x}F_2}$$

$$\frac{\partial z}{\partial y} = -\frac{\varphi_y}{\varphi_z}$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

(4) 设$F(x+\frac{z}{y}, y+\frac{z}{x})=0$, $F$可微，求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, dz$.

解法二：方程两端求对$x$的偏导数，有

$$F_1'(1+\frac{1}{y}\cdot\frac{\partial z}{\partial x})+F_2'(-\frac{z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x})=0$$

解得 $$\frac{\partial z}{\partial x}=\frac{F_2'\cdot\frac{z}{x^2}-F_1'}{\frac{1}{y}F_1'+\frac{1}{x}F_2'}$$

方程两端求对$y$的偏导数，有 $$\frac{\partial z}{\partial y}=\frac{F_1'\cdot(\frac{z}{y^2})-F_2'}{\frac{1}{x}F_2'+\frac{1}{y}F_1'}$$

(4) 设$F(x+\frac{z}{y}, y+\frac{z}{x})=0$, $F$可微，求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $dz$.

解法三：利用全微分形式的不变性求偏导

左端是对$\varphi(x, y, z)=F(x+\frac{z}{y}, y+\frac{z}{x})$求微分

$$F_1 d(x+\frac{z}{y})+F_2 d(y+\frac{z}{x})=0$$

$$F_1(dx+\frac{ydz-zdy}{y^2})+F_2(dy+\frac{xdz-zdx}{x^2})=0$$

$$(\frac{1}{y}F_1+\frac{1}{x}F_2)dz=(-F_1+F_2\cdot\frac{z}{x^2})dx+(-F_2+F_1\cdot\frac{z}{y^2})dy$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

(5)设$u=u(x,y)$由下列方程组确定 $\begin{cases} u=f(x,y,z,t) \\ g(y,z,t)=0 \\ h(z,t)=0 \end{cases}$

$f,g,h$可微，$g_z h_t - g_t h_z \neq 0$, 求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}$

解 方程组确定：$z=z(x,y),u=u(x,y),t=t(x,y)$

方程组两边对$x$求导：

$$\begin{cases} \dfrac{\partial u}{\partial x}=f_x+f_z\dfrac{\partial z}{\partial x}+f_t\dfrac{\partial t}{\partial x} \\ g_z\dfrac{\partial z}{\partial x}+g_t\dfrac{\partial t}{\partial x}=0 \\ h_z\dfrac{\partial z}{\partial x}+h_t\dfrac{\partial t}{\partial x}=0 \end{cases}$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial t}{\partial x}=0$$

$$\Rightarrow \frac{\partial u}{\partial x}=f_x$$

方程组两边对 $y$求导：

$$\begin{cases} u = f(x, y, z, t) \\ g(y, z, t) = 0 \\ h(z, t) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial y} = f_y + f_z \dfrac{\partial z}{\partial y} + f_t \dfrac{\partial t}{\partial y} \\ g_z \dfrac{\partial z}{\partial y} + g_t \dfrac{\partial t}{\partial y} = -g_y \\ h_z \dfrac{\partial z}{\partial y} + h_t \dfrac{\partial t}{\partial y} = 0 \end{cases}$$

$$\frac{\partial z}{\partial y} = \frac{-g_y h_t}{g_z h_t - g_t h_z}$$

$$\frac{\partial t}{\partial y} = \frac{g_y h_z}{g_z h_t - g_t h_z}$$

$$\frac{\partial u}{\partial y} = f_y - \frac{g_y h_t f_z - g_y h_z f_t}{g_z h_t - g_t h_z}$$

# 7.5 - 7.7 内容及要求

补充：当曲线由交面式方程给出时求空间曲线的切线及法平面

$$\Gamma:\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}\quad(1)\qquad\Rightarrow\begin{cases}x=x\\y=\varphi(x)\\z=\psi(x)\end{cases}$$

点$M_0(x_0,\ y_0,\ z_0)$是Γ 上一点

（1）式等于两端对$x$求导数　解出 $\dfrac{dy}{dx},\dfrac{dz}{dx}$

切向量为：$\vec{T}=\{1,\varphi'(x_0),\psi'(x_0)\}$

切线方程为

$$\frac{x-x_0}{1}=\frac{y-y_0}{\varphi'(x_0)}=\frac{z-z_0}{\psi'(x_0)},$$

法平面方程为

$$1\cdot(x-x_0)+\varphi'(x_0)(y-y_0)+\psi'(x_0)(z-z_0)=0.$$

设$\Gamma$: $\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0. \end{cases}$

交面式空间曲线的切向量的另一求法：

$$\vec{n}_1=\{F_x(M_0),F_y(M_0),F_z(M_0)\},$$

$$\vec{n}_2=\{G_x(M_0),G_y(M_0),G_z(M_0)\},$$

切线为两切平面的交线， 切向量$\mathrm{T}\,/\!/\,\mathrm{n}_1\times\mathrm{n}_2$.

$$\vec{\mathrm{T}}=\begin{vmatrix} i & j & k \\ F_x(M_0) & F_y(M_0) & F_z(M_0) \\ G_x(M_0) & G_y(M_0) & G_z(M_0) \end{vmatrix}$$

**例1** 在曲面$z=xy$上求一点，使这点处的法线垂直于平面$x+3y+z+9=0$，并写出这法线的方程。

解：曲面$z=xy$上点($x_0$，$y_0$，$z_0$)处的一个法向量为

$$\vec{n}=\{f_x(x_0,y_0),f_y(x_0,y_0),-1\}=\{y_0,x_0,-1\}$$

已知平面的法向量为$n_1=\{1, 3, 1\}$，依题意应有$n // n_1$，即

$$\frac{y_0}{1}=\frac{x_0}{3}=-1$$

故所求点为(－3，－1，3)，所求法线方程为

$$\frac{x+3}{1}=\frac{y+1}{3}=\frac{z-3}{1}.$$

例4 求函数$z=1-(\frac{x^2}{a^2}+\frac{y^2}{b^2})(a>0,b>0)$在

点$(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}})$处沿椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$的内法线

方向的方向导数。

解：曲线$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 在$M$点的法向量为

$$\vec{n}=\pm\left\{\frac{2x}{a^2},\frac{2y}{b^2}\right\}_M=\pm\left\{\frac{\sqrt{2}}{a},\frac{\sqrt{2}}{b}\right\},$$

$\because$内法线

$\therefore$取负号！

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曲线方程为 $F(x,y)=0$

法向量：$\pm\{F_x(x_0,y_0),F_y(x_0,y_0)\}$

**例4** 求函数$z=1-(\frac{x^2}{a^2}+\frac{y^2}{b^2})(a>0,b>0)$在点$(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}})$ 处沿椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$的内法线方向的方向导数。

解：曲线$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 在$M$点的内法线方向为

$$\vec{n}=\left\{-\frac{2x}{a^2},-\frac{2y}{b^2}\right\}_M=\left\{-\frac{\sqrt{2}}{a},-\frac{\sqrt{2}}{b}\right\},$$

又因为 $\text{gard}z\big|_M=\left\{\frac{-2x}{a^2},\frac{-2y}{b^2}\right\}_M=\left\{-\frac{\sqrt{2}}{a},-\frac{\sqrt{2}}{b}\right\}=l$

所以 $\left.\frac{\partial z}{\partial l}\right|_M=\mathbf{grad}z\,|_M\cdot n^\circ=\frac{l\cdot l}{|l|}=|l|=\sqrt{\frac{2}{a^2}+\frac{2}{b^2}}$

**例 5** 在第一卦限内作椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$的切平面，使切平面与三个坐标面所围成的四面体体积最小，求切点坐标.

**解** 设$P(x_0, y_0, z_0)$为椭球面上一点,

令$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$,

则$F'_x|_P=\frac{2x_0}{a^2}$，　$F'_y|_P=\frac{2y_0}{b^2}$，　$F'_z|_P=\frac{2z_0}{c^2}$

过$P(x_0, y_0, z_0)$的切平面方程为

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)+\frac{z_0}{c^2}(z-z_0)=0,$$

**例 5** 在第一卦限内作椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$的切平面，使切平面与三个坐标面所围成的四面体体积最小，求切点坐标.

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)+\frac{z_0}{c^2}(z-z_0)=0,$$

化简为 $\frac{x\cdot x_0}{a^2}+\frac{y\cdot y_0}{b^2}+\frac{z\cdot z_0}{c^2}=1,$

切平面在三个轴上的截距： $x=\frac{a^2}{x_0}$， $y=\frac{b^2}{y_0}$， $z=\frac{c^2}{z_0}$

所围四面体的体积 $V=\frac{1}{6}xyz=\frac{a^2b^2c^2}{6x_0y_0z_0},$

求$V=\dfrac{a^2b^2c^2}{6xyz}$的最小值，转化为求 $u=xyz$的最大值，

即$\ln u=\ln x+\ln y+\ln z$的最大值

$$\text{条件}:\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$G(x,y,z)=\ln x+\ln y+\ln z+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\text{由}\begin{cases} G'_x=0, \quad G'_y=0, \quad G'_z=0 \\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{y^2}{c^2}-1=0 \end{cases},$$

即
$$\begin{cases} \dfrac{1}{x}+\dfrac{2\lambda x}{a^2}=0 \\ \dfrac{1}{y}+\dfrac{2\lambda y}{b^2}=0 \\ \dfrac{1}{z}+\dfrac{2\lambda z}{c^2}=0 \\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1=0 \end{cases}$$

可得
$$\begin{cases} x=\dfrac{a}{\sqrt{3}} \\ y=\dfrac{b}{\sqrt{3}} \\ z=\dfrac{c}{\sqrt{3}} \end{cases}$$

当切点坐标为 $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}}\right)$时,

四面体的体积最小$V_{\min}=\dfrac{\sqrt{3}}{2}abc.$

**例 6** 求旋转抛物面 $z = x^2 + y^2$ 与平面 $x + y - 2z = 2$ 之间的最短距离.

**解** 设 $P(x, y, z)$ 为抛物面 $z = x^2 + y^2$ 上任一点,则 $P$ 到平面 $x + y - 2z - 2 = 0$ 的距离为 $d$,

$$d = \frac{|x + y - 2z - 2|}{\sqrt{1 + 1 + (-2)^2}} = \frac{1}{\sqrt{6}}|x + y - 2z - 2|$$

分析: 本题变为求一点 $P(x, y, z)$, 使得 $x, y, z$ 满足 $x^2 + y^2 - z = 0$ 且使 $d = \frac{1}{\sqrt{6}}|x + y - 2z - 2|$

(即 $d^2 = \frac{1}{6}(x + y - 2z - 2)^2$) 最小.

令 $F(x,y,z)=\frac{1}{6}(x+y-2z-2)^2+\lambda(z-x^2-y^2)$,

$$\begin{cases} F'_x=\frac{1}{3}(x+y-2z-2)-2\lambda x=0, & (1) \\ F'_y=\frac{1}{3}(x+y-2z-2)-2\lambda y=0, & (2) \\ F'_z=\frac{1}{3}(x+y-2z-2)(-2)+\lambda=0, & (3) \\ z=x^2+y^2, & (4) \end{cases}$$

解此方程组得 $x=\frac{1}{4}, y=\frac{1}{4}, z=\frac{1}{8}$.

例 6 求旋转抛物面$z = x^2 + y^2$与平面$x + y - 2z = 2$之间的最短距离.

得唯一驻点 $(\frac{1}{4}, \frac{1}{4}, \frac{1}{8})$,

根据题意距离的最小值一定存在，且有唯一驻点，故必在$(\frac{1}{4}, \frac{1}{4}, \frac{1}{8})$处取得最小值.

$$d_{\min} = \frac{1}{\sqrt{6}}\left|\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - 2\right| = \frac{7}{4\sqrt{6}}.$$

**例 6** 求旋转抛物面 $z = x^2 + y^2$ 与平面 $x + y - 2z = 2$ 之间的最短距离.

解法二：作切平面平行于平面，设切点为 $(x_0, y_0, z_0)$

切点到平面的距离即为旋转抛物面与平面之间的最短距离

则 $\vec{n} = \{2x_0, 2y_0, -1\} // \{1, 1, -2\}$

$$\Rightarrow \quad x = \frac{1}{4}, y = \frac{1}{4}, z = \frac{1}{8}.$$

$$d_{\min} = \frac{1}{\sqrt{6}}\left|\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - 2\right| = \frac{7}{4\sqrt{6}}.$$